PRACTICAL MATHEMATICS AND THEORETICAL MATHEMATICS

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Abstract

This paper explores the role of both theoretical and practical mathematics in university curricula. While math instruction often focuses on theoretical reasoning, it is equally important to successfully integrate both kinds of knowledge. As the landscape of mathematics education changes and the demand for practical mathematical skills increases across various fields, this discussion is especially timely. This paper begins by examining Sierpińska's *pressing challenge* regarding the fragility of theoretical thinking and situates this issue within a broader intellectual context. We incorporate views from Aristotle, Poincaré, Sierpińska, Tall, and Freudenthal, each representing a unique tradition regarding the relationship between practical and theoretical knowledge. Additionally, the paper references findings from collaborative research with mathematicians to connect theoretical insights to current teaching practices. The goal is to show how these perspectives, combined with research evidence, can inform teaching methods, curriculum development, and classroom approaches, opening an exciting area of exploration. Its contribution lies in fostering a dialogue that needs further research, especially on how students can transition smoothly between practical and theoretical mathematics.

Keywords: Sierpińska, Tall, Freudenthal, Aristotle, Theoretical mathematics, Practical mathematics

INTRODUCTION

Balancing the theoretical and practical parts of mathematics teaching remains a major challenge for educators. Math instructors often can find themselves apologizing to students for focusing too much on abstract theory, knowing that such material can overwhelm learners. To keep students engaged, they often reassure them that concrete examples, typically involving some computations, will follow. This internal decision-making is shaped by multiple factors, including students' comfort with math and their previous experiences with mathematical ideas.

Drawing from her studies of students' struggles with linear algebra, Sierpińska (2000; 2004) issued a pressing challenge that continues to resonate in mathematics education. She observed that many students are more comfortable with the practical aspects of the subject, especially computation, and "find reading mathematics difficult and tend to limit their reading to examples and exercises" (2000, p. 244). This resistance to theory, she argued, cannot simply be ignored or minimized, and it requires sustained attention to pedagogy. Sierpińska called for teaching approaches that utilize applications and students' computational inclinations as entry points into the subject. Yet she also insisted that theoretical thinking is indispensable in the long run.

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As she put it:

> In the long run, we still want our students to be able and willing to engage in theoretical thinking. Maybe there is just no scientific knowledge without theoretical thinking, but a blind application of techniques, or, at most, a technology without reflection on its relevance and possible consequences. In this case, if education has anything to do with scientific knowledge, then, yes, something has to be done.… Theoretical thinking needs special nurturing; nature and everyday socialization do not suffice (2004, p. 245).

By framing theoretical thinking as something fragile and requiring deliberate cultivation, Sierpińska asks educators to confront a difficult truth: without intentional support, students may never move beyond the practical and computational.

Starting with this pressing challenge, this paper engages with other influential perspectives by referencing the work of Aristotle, Poincaré, Sierpińska, Tall, and Freudenthal. Their ideas, combined, offer a multi-layered view that can help in understanding and addressing the ongoing tension between practice and theory in university mathematics education. To explore the relationship between practical and theoretical mathematics, this paper intentionally draws on a group of thinkers from philosophy, mathematics, and mathematics education. Aristotle (*Metaphysics,* trans. Reeve, 2016) provides the classical philosophical foundations for distinguishing different types of knowledge. Poincaré (1913), working at the intersection of mathematics and physics, demonstrates how theory and application influence each other. Sierpińska (2004) presents a model of practical and theoretical thinking in the context of mathematics education. Freudenthal (1968) grounds mathematics in human activity, emphasizing its usefulness and the need for reinvention. Tall (2013) offers a developmental model of mathematical thinking that connects practical, theoretical, and axiomatic mathematics. Collectively, these perspectives span historical and disciplinary boundaries, providing a multifaceted lens for understanding mathematics education.

In bringing these perspectives into dialogue, it is important to clarify what is meant by "theory." In this paper, the main focus is on mathematical theory: the abstract, general, and deductive frameworks that set mathematics apart from its more immediate, practical applications. This is the sense in which Aristotle distinguished explanation from experience, or Poincaré described theory as preceding and guiding experimental discovery. At the same time, we also reference theories in mathematics education—such as Sierpińska's (2000; 2004) account of practical and theoretical thinking or Tall's three-worlds framework—which offer conceptual tools for analyzing pedagogy. These educational theories demonstrate that the interaction between practical and theoretical mathematics occurs in the context of teaching and learning.

OBJECTIVES

The objective of this paper is to analyze the interplay between practical and theoretical mathematics from multiple perspectives, with a particular focus on their implications for university teaching and learning. Building on Sierpińska's (2000; 2004) pressing challenge, the paper seeks to clarify how both forms of knowledge can be supported in the classroom, not in opposition but in complementarity.

To pursue this goal, the paper engages with ideas from Aristotle, Poincaré, Freudenthal, Sierpińska, and Tall. These thinkers represent different perspectives—philosophy, mathematics, and mathematics education—that, when brought into conversation, reveal both similarities and useful tensions. Their contributions are not examined in full detail, but they serve as resources for reconsidering the balance between theory and practice in today's mathematics education. What makes this path particularly exciting is the opportunity to unite voices that are seldom seen together, thereby opening up new possibilities for teaching and curriculum development. In today's world, both mathematical theory and applied mathematics are essential, and educators need to be ready to find meaningful and practical ways to incorporate these aspects into the classroom.

THE STRUCTURE OF THE PAPER

This paper adopts a qualitative, interpretive approach (Creswell & Poth, 2018; Wolcott, 1994), drawing on philosophical texts, historical reflections, and research in mathematics education. The analysis is comparative, exploring how different perspectives—philosophical, mathematical, and educational—frame the relationship between practical and theoretical knowledge.

Instead of using each thinker's entire intellectual system, the focus is on extracting and contextualizing specific concepts: Aristotle's explanation of causes and science, Poincaré's thoughts on analysis and physics, Freudenthal's call for usefulness, Sierpińska's description of practical and theoretical thinking, and Tall's framework of practical, theoretical, and axiomatic mathematics. These ideas are then analyzed in relation to current challenges in teaching university mathematics. The approach is pedagogical, considering each perspective not only in its original context but also regarding its implications for today's mathematics educators. While the analysis draws from diverse sources, the goal is not exhaustive coverage but an exploratory dialogue across traditions. In this way, the paper serves as a starting point, opening a line of inquiry rather than drawing final conclusions.

The question of how practical and theoretical mathematics connect has long been a topic of interest. It has been discussed in philosophy, scientific practice, and mathematics education, yet remains unresolved in university classrooms. Students are expected to develop both practical skills and the ability to think abstractly, but figuring out how to combine these aspects in teaching is not straightforward and is rarely addressed.

ARISTOTLE – EXPERIENCE, CRAFT, AND SCIENTIFIC KNOWLEDGE

The distinction between practical and theoretical knowledge receives one of its earliest and most systematic treatments in Aristotle's *Metaphysics*. At the very opening of Book Alpha (A1), he develops a hierarchy of kinds of knowledge, ranging from the most basic perceptual awareness to the most elevated form of theoretical wisdom (*sophia*). The following passage is from *Metaphysics* A1 (*Reeve, 2016, pp. 2–4*).

Aristotle begins by stressing the foundational role of lived experience: "experience seems pretty much similar to scientific knowledge and craft knowledge. But scientific knowledge and craft knowledge come to humans through experience. For experience made craft as Polus says and lack of experience, luck". Experience, for Aristotle, is the raw material from which more systematic forms of knowing are generated. From repeated encounters, human beings are able to generalize. As he explains, "craft knowledge comes about when, from many intelligible objects belonging to experience, one universal supposition about similar things come about". This marks the move from particular observations to universal principles. Yet in practice, success is not always on the side of the theoretical knower. Aristotle observes:

> With a view to action, then, experience seems no different from craft knowledge – on the contrary, we even see experienced people being more successful than those who have an account but are without experience. The cause of this is that experience is knowledge of particulars, whereas craft knowledge is of universals, and actions and productions are all concerned with particulars.

Here he underscores the practical value of particulars, even as he ranks universals more highly. The real distinction, however, lies in explanatory power.

> We regard knowledge and comprehension as characteristic of craft rather than the experience, and take it that the craftsmen are wiser than experienced people, on the supposition that in every case wisdom follows along rather with knowledge than experience. This is because craftsmen know the cause, whereas the experienced people do not. For experienced people know the that but do not know the why, whereas craftsmen know the why, that is, the cause.

Thus, causal understanding elevates craft above mere experience. Teaching provides further confirmation of this hierarchy:

> On the whole too an indication of the person who knows, as opposed to the person who does not know, in his capacity to teach. That is why we think craft knowledge to be more like scientific knowledge than experience is, since craftsmen can teach, while experienced people cannot.

For Aristotle, true knowledge is transmissible because it rests on causes and principles rather than isolated facts. Even so, perceptual awareness, though essential, is limited:

> We do not think that any perceptual capacities whatsoever constitute wisdom, even though they are most in control, at any rate, of the knowledge of particulars. Still they do not tell us the why of anything (for example, why fire is hot), but only that it is hot.

Perception delivers the data of experience, but not the explanatory depth of knowledge. Finally, Aristotle equates wisdom (*sophia*) with the pursuit of ultimate causes.

> Everyone takes what is called "wisdom" to be concerned with the primary causes and the starting-

> points. And so, as we said earlier, the person of experience seems to be wiser than those who have any perceptual capacity whatsoever, a craftsman than experienced people, an architectonic craftsman than a handicraftsman, and the theoretical sciences than productive ones. So it is clear that theoretical wisdom is scientific knowledge of certain sorts of starting-points and causes.

In this final statement, he places the highest form of knowledge not in perception, experience, or craft, but in their theoretical understanding of first principles.

As Michael Detlefsen (2005), a philosopher with expertise in mathematics and history, explains, "the question was what cause might come to in the case of mathematical objects. To help answer this question, Aristotle famously distinguished four notions of cause – formal, material, efficient, and final" (p. 241). Aristotle himself writes:

> Evidently we have to acquire knowledge of the original causes (for we say we know each thing only when we think we recognize its first cause), and causes are spoken of in four senses. In one of these we mean the substance, i.e. the essence (for the 'why' is referred finally to the formula [*logos*], and the ultimate 'why' is a cause and principle [arche]; in another, the matter or substratum; in a third the source of the change; and in the fourth the cause opposed to this, that for the sake of which and the good (for this is the end of all generation and change)" *(Metaphysics I, 3, 983a25–31)*.

According to Detlefsen, "Aristotle believed that it was the first or formal notion of cause that applied to mathematical knowledge" (p. 241). The formal cause does not depend on motion but reflects the inherent nature of the object itself. In Detlefsen's view (2005, p. 241):

> One could thus acquire epistēmē of a mathematical object by knowing its formal cause – that is, its nature or essence. The nature or essence of an object, however, was supposed to be given by a proper definition of that object. Grasp of the nature of a mathematical object should therefore consist in grasp of its proper definition. From this it follows that, at bottom, knowledge of cause in mathematics consists in knowledge of a definition. As Aristotle so succinctly put it: The "why" is referred ultimately … in mathematics … to the "what" (to the definition of straight line or commensurable or the like) (*Physics, Rev.], II, 7, 198a16–18*).

Heath (1980), in *Mathematics in Aristotle*, argued that Aristotle engaged deeply with mathematical ideas, even if his purpose was largely philosophical. Aristotle discussed topics such as infinity, continuity, proportion, and geometry, often drawing on the work of earlier mathematicians like Euclid and Eudoxus.

POINCARÉ – ANALYSIS AND PHYSICS

Henri Poincaré, who worked in both mathematics and physics, reflected deeply on the relationship between analysis and experimental science. He emphasized that the analyst does not replace experiment but gives it conceptual clarity:

> The physicist can not ask of the analyst to reveal to him a new truth; the latter could at most only aid him to foresee it. It is a long time since one still dreamt of forestalling experiment, or of constructing the entire world on certain premature hypotheses.… All laws are therefore deduced from experiment; but to enunciate them, a special language is needful; ordinary language is too poor, it is besides too vague, to express relations so delicate. So rich, and so precise" (Poincaré, 1913, p. 133).

This interplay is vividly illustrated in Fourier's invention of series, developed to solve a problem of heat propagation in physics:

> Fourier's series is a precious instrument of which analysis makes continual use, it is by this means that it has been able to represent discontinuous functions; Fourier invented it to solve a problem of physics relative to the propagation of heat. If this problem had not come up naturally, we should never have dared to give discontinuity its rights; we should still long have regarded continuous functions as the only true functions (p. 136).

A concrete physical challenge thus pushed mathematicians to confront discontinuity, opening the way to bold new ideas in analysis.
Poincaré then turned to the opposite case, where mathematics itself reached further than experiment. Maxwell's theory of electromagnetism anticipated confirmation by twenty years, and Poincaré captured the source of this achievement:

> Maxwell was twenty years ahead of experiment. How was this triumph obtained? It was because Maxwell was profoundly steeped in the sense of mathematical symmetry; would he have been so, if others before him had not studied this symmetry for its own beauty? (p. 134).

Where Fourier showed how physical need can generate new mathematics, Maxwell showed how mathematics, pursued for its own beauty, can lead science into entirely new domains.
Beyond many examples that Poincaré shared, he also spoke directly about the aims of mathematics itself. He rejected a purely practical view, insisting that:

> a science made solely in view of applications is impossible; truths are fecund only if bound together. If we devote ourselves solely to those truths whence we expect an immediate result, the intermediary links are wanting and there will no longer be a chain (p. 279).

For him, mathematics had a threefold purpose:

> It must furnish an instrument for the study of nature. But that is not all: it has a philosophic aim and, I dare maintain, an esthetic aim.… Mathematics deserves to be cultivated for its own sake, and the

> theories are inapplicable to physics as well as the others. Even if the physical aim and the esthetic aim were not united, we ought not to sacrifice either (p. 280).

At the same time, Poincaré never lost sight of nature's complexity:

> However varied may be the imagination of man, nature is still a thousand times richer. To follow her we must take ways we have neglected, and these paths lead us often to summits whence we discover new countries. What could be more useful (p. 135).

In this vision, mathematics moves between usefulness, beauty, and philosophy—sometimes following nature's demands, sometimes leading it, but always enlarging our horizons.

SIERPIŃSKA'S MODEL OF PRACTICAL AND THEORETICAL THINKING

Anna Sierpińska (2000; 2004) explored practical and theoretical thinking in students' challenges with understanding linear algebra. She (2004, p. 6) defined practical and theoretical thinking as:

> In our research, mathematical thinking was assumed to be based on an interaction of practical and theoretical thinking. Practical thinking was considered to be the source of wonder and curiosity, leading to bold conjectures, which then provided food for theoretical thought. Our definition of theoretical thinking was, therefore, based on a number of oppositions related to reasons for thinking; objects of thinking; means of thought, main concerns; products of thinking, which distinguished it from practical thinking.

Sierpińska (2000, p. 244) sounded the alarm and claimed that:

> In our research on the teaching of linear algebra we found that no matter how we tried to approach the content, students' difficulties seemed to persist. The reason could be that we introduced all sorts of changes but never gave up teaching the structural theory of linear algebra. It is not enough to just make the structural content more concrete through working in low dimensions and using visualizations. In fact, visualizations themselves are problematic; they may lead to irrelevant interpretations which make the understanding more, not less difficult (Sierpińska, Dreyfus, and Hillel, 1999). It is possible that much more radical changes of the content of teaching have to be introduced.

In her view, "the inadequacy of the curriculum is only a part of the problems of the ineffectiveness of our teaching of linear algebra" (Sierpińska, 2000, p. 245). She noted that the broader issue is the ineffective teaching methods at the university level, which create a passive and authoritarian learning environment. This can lead students to view theory as irrelevant and focus solely on surviving academic tasks rather than developing scientific knowledge. The authoritarian learning environment may also discourage students from owning the theories and incorporating them into their own problem-solving efforts.

Research and teaching experience suggest that many students are more comfortable with the practical aspects of the subject, particularly when computations are involved. “They find reading mathematics difficult and tend to limit their reading to examples and exercises” (Sierpinska, 2000, p. 244). This disparity calls for a more “practical” approach, where linear algebra is taught through applications rather than explicit theoretical definitions. This could help students engage with the subject in a more accessible way, using the tools of linear algebra to solve real-world problems.

In a case study, Sierpińska et al. (2002, cited in Sierpinska, 2004) interviewed 14 high-achieving students of linear algebra. The results revealed that the students tended to gravitate toward practical over theoretical thinking. Sierpińska (2004) stated that:

> Examples of students’ work in linear algebra will show that successful students of university level mathematics certainly have a sense of what it means to work within a theoretical system. But they are also very “practical” in moving about the theory, capitalizing on previous experience without trying to simply reproduce a method learned by heart, and picking exactly what is needed from the theory to get straight to the solution. They are ready to give up on rigor, consistency of notation, generality of the solution, if this is not absolutely necessary for obtaining a satisfactory solution. They solve the problem at hand; they do not develop a theory of solving all problems of a kind, yet there is an undercurrent of generalizable techniques in their solutions. Perhaps this could be classified as a kind of “situated knowledge”. This is why analyzing their solutions requires getting into the mathematical content of the problem. And it is the content specificity of these analyses that, I think, makes them useful for didactics of mathematics. (pp. 1-2)

To work toward correcting this misalignment and instead ‘foster theoretical thinking’ (p. 5) in her first-year linear algebra courses, Sierpińska (2004) administered two weekly quizzes, each comprising two short conceptual questions that required theoretical thinking. The analysis of students’ solutions showed that “those who obtained correct solutions were not only ‘theoretical thinkers’; they also had developed a ‘practical understanding of theory’” (pp. 5-6), which she described as necessary for finding clever solutions to problems. Theoretical thinking alone, she says, is not sufficient for solving novel problems.

She also declared that, “teaching the theory of mathematics is not a problem. The problem is how to teach the art of mathematics. The art of doing mathematics is a complex thing and there is little systematic, yet content specific, knowledge about it” (p. 2). By the art of mathematics in mathematics education, she meant, “thinking involved in solving more or less conventional or imaginative, closed or open, formal or informal exercises and problems” (p. 3).

Sierpińska (2000) cautioned that “Theoretical thinking needs special nurturing; nature and everyday socialization do not suffice” (p. 245). This shows that more deliberate strategies are necessary to promote theoretical thinking; otherwise, they will fall short. Sierpińska’s (2004, p. 7) model distinguishes between practical and theoretical thinking. She described her model as “a very crude model of an extremely complex reality” (2004, p. 4) and “more refined models are needed for the purposes of didactics, and even these could be uninformative unless complemented by epistemological analyses of the particular mathematical subject matter" (p. 1). This suggests that her model is best viewed as a strong starting point, useful for framing the problem and providing terminology that guides the conversation, exemplifying the notions and more; however, it requires further development.

TALL'S MODEL OF THEORETICAL MATHEMATICS AND PRACTICAL MATHEMATICS

David Tall (2013) organized his book, *How Humans Think Mathematically*, to show how mathematics is categorized into three levels: 1) Practical mathematics, 2) Theoretical mathematics, and 3) Axiomatic formal mathematics.

Practical mathematics involves hands-on experiences with shapes, space, and arithmetic, where children learn to recognize properties without exploring their deeper implications. Theoretical mathematics builds on this foundation by using definitions and properties to deduce new relationships and engage in deductive reasoning. Axiomatic formal mathematics emphasizes rigorous definitions and proofs of abstract concepts, such as proving laws like commutativity through formal methods. Tall highlights that these levels are interconnected and essential for developing mathematical literacy, emphasizing the role of educators in supporting students through these transitions (see Figure 1). In Tall's (2023) view, Figure 1 "distils the broad outline of development of mathematical thinking through practical, theoretical and axiomatic formal levels of development in three worlds of mathematics" (p. 226).

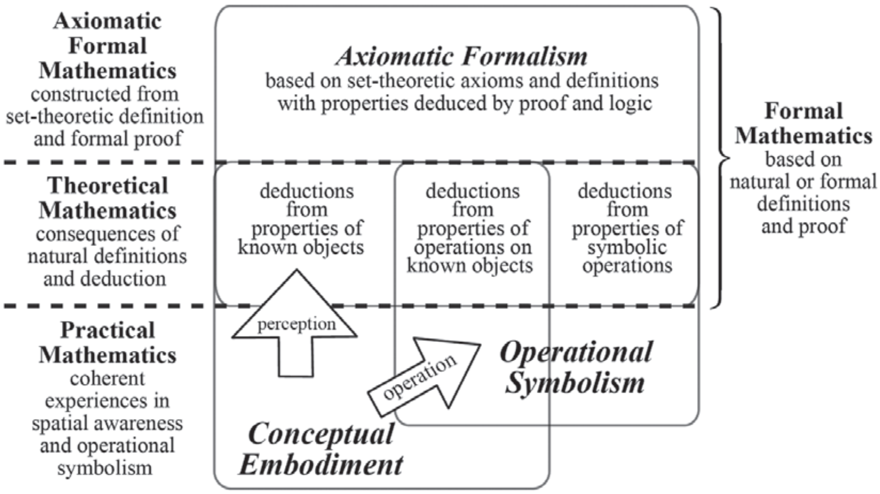

Figure 1. Long-term development of mathematical thinking (based on Tall, 2013, p. 403), cited in Tall, 2023, p. 226

Tall (2013) formulated three strands of mathematical development: namely, *conceptual embodiment, operational symbolism, and axiomatic formalism.* These strands focus on physical and mental objects and their proper as well as operations on objects and formally defined properties, respectively. More recently, Tall (2023) introduced *a meaningful long-term framework for mathematical thinking* based on his notion of 'three worlds of mathematical thinking', each operating in a different way: The '*embodied*' based on perception through the senses with physical and mental constructs; the '*symbolic',* where expressions such as $3x^2+4y$ may be conceived dually as processes to be performed or as mental objects to be manipulated; and the '*formal*', based on set-theoretic definition and deduction (Tall, 2013). An example of this framework in practice may be seen in a (finite-dimensional) vector space where: An '*embodied*' interpretation of a vector is given as an arrow in space representing magnitude and direction. A 'symbolic' interpretation is given as an *n*-tuple of coordinates $(x_1, x_2, \ldots, x_n)$ over a field. A '*formal*' interpretation is based on set-theoretic axioms for a vector space and formal definitions of specific properties of vectors (Tall, 2013). In his view, "these develop long-term from practical mathematics interacting with the world we live in, to theoretical mathematics used in society to model and predict outcomes, and axiomatic formal mathematics using axiomatic definitions and formal proof" (Tall, 2023, p. 218).

The extended framework "includes other social and personal aspects related to the nature of mathematics and how it is conceived and interpreted by us as human beings" (Tall, 2023, p. 248). Individuals living in a complex world with different backgrounds and needs require "different individuals to use different kinds of mathematics in productive ways" (p. 219). The meaningful long-term framework for mathematical thinking (Tall, 2023) captures and highlights many aspects of mathematics and its teaching and learning. The ideas expressed in the framework focus the readers' attention on long-term effects and making sense of mathematics. Tall's (2023) intentions were "to encourage learners to develop a personal sense of confidence by realising that it is not their inadequacy that causes them to have difficulty in understanding more sophisticated ideas, it also relates to the changing nature of mathematics itself" (p. 249). This generalizes to a multi-contextual overview that encompasses the long-term historical evolution of mathematical thinking, the long-term development of mathematical thinking in an individual over a lifetime, and the corporate evolution of ideas within a socially shared community of practice. The essence of formulating the framework is to offer:

> … a long-term framework for the meaningful development of mathematical thinking that takes into account the increasing sophistication of mathematical ideas and the cognitive and emotional growth of the individual. It also offers a contextual overview to encourage the comparison and cooperation of different communities of practice. It does not predict the future. It offers a framework for readers to challenge their own beliefs to make informed choices (Tall, 2023, p. 251).

Tall asserts that there is a place for both practical and theoretical mathematics. "We live in a complex society that requires different individuals to use different kinds of mathematics in productive ways" (Tall, 2023, p. 219). These range from requiring practical mathematics for everyday settings to different professions that may require technical mathematics, while others may need more theoretical mathematics that enables them to model real-life situations and predict possible outcomes. Some may go on to explore more formal aspects of pure mathematics and logic, involving set-theoretic axioms, definitions, and formal proofs (Tall, 2023).

According to the differing requirements and usage of mathematics, the framework of "meaningful long-term for mathematical thinking that encompasses the many different aspects that are essential in a complex society" (p. 248). The long-term development incorporates the three-world framework of embodiment, symbolism, and formalism through practical, theoretical, and axiomatic formal mathematics (see figure 1). He defines practical mathematics as "the coherent experiences in spatial awareness and operational symbolism" (p. 226), and theoretical mathematics as "consequences of natural definitions and deductions" (p. 226). His goal was "to reflect on this overall picture, to compare the coherence of practical mathematics with the consequence of definition and deduction in theoretical mathematics, and with set-theoretic definition and deduction in axiomatic formal mathematics" (p. 244). He emphasizes that "the framework does not represent a strict sequential development …*all aspects in the framework may be relevant in any order* (pp. 226 -227).

Tall (2023) provided examples of practical and theoretical mathematics in each of the worlds. A sample of the examples in Table 1 not only highlights the characteristics of the world they represent but also reveals insights about the mathematics and its evolution from practical to theoretical.

Table 1. Examples of practical and theoretical mathematics

	Practical Mathematics	**Theoretical Mathematics**
Embodied world	"involves not only perception and operation with physical objects but also imaginative thinking related to coherent recognition and description of properties of objects, such as those arising in ruler and compass constructions in Euclidean geometry" (p. 227).	"involves definition and proof using carefully chosen definitions of naturally occurring objects and properties that follow as a consequence of a deductive argument" (p. 227).
Arithmetic and algebra	"involves observed recognition of operations in arithmetic, such as the fact that adding a list of numbers gives the same total regardless of the order of addition" (p. 227).	"selects specific rules such as the commutative, associative and distributive laws of addition and subtraction in their simplest forms and deduces general properties. This involves minimal definitions but requires more sophisticated proofs" (p. 227).

Using induction as an example, Tall (2023) revealed a significant distinction between all three levels of thinking:

> In axiomatic formal mathematics, induction uses the Peano postulates and takes the form of a finite proof: prove the first stage, then prove the general deduction that if it is true at one stage, it is true at the next, then quote the induction axiom that asserts the truth of all stages. Dealing with the infinite reveals a crucial distinction between practical, theoretical and axiomatic formal levels of thinking (p. 227).

This example on induction highlights the importance of practical, theoretical, and formal mathematics, demonstrating how each level uniquely contributes to the development of mathematical thought.

HANS FREUDENTHAL – TEACHING MATHEMATICS SO AS TO BE USEFUL

Hans Freudenthal (1968) opened his address with a striking declaration: "I will not speak about how to teach mathematics so as to be useful but about why we should teach mathematics so as to be useful, or rather about why we should teach mathematics so as to be more useful" (p. 3). His concern was not with techniques, but with the deeper justification for usefulness as a guiding principle in mathematics education. He observed that while much research had focused on learning in controlled settings, very little was known about how individuals actually apply what they have learned. This gap explained why "most people never succeed in putting their theoretical knowledge to practical use" (p. 4). Yet, he argued, mathematics had become indispensable for both the physical and social worlds, and was "needed not by a few people, but virtually by everybody" (p. 5). The dilemma was that the most abstract mathematics is also the most flexible and powerful in principle, but only if learners can make it their own. Freudenthal sharply rejected two extremes: teaching pure mathematics with the hope that students would later apply it, or teaching narrow "useful mathematics" locked into fixed contexts. Neither worked. As he put it:

> Between two extreme attitudes one may be inclined to try compromising. If this means teaching pure mathematics and afterwards to show how to apply it, I am afraid we are no better off. I think this is just the wrong order. (p. 5).

His example contrasted the success of arithmetic, which is patiently introduced through recurring concrete contexts, with the failure of fractions, too often taught in abstraction with only ceremonial links to experience. For Freudenthal (1968), the answer was in returning mathematics to its core as an activity:

> Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing, not its result. Its result is a system, a beautiful closed system, closed, with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics (p. 7).

His passion for *mathematizing* became the cornerstone of Freudenthal's later development of Realistic Mathematics Education (RME). He believed that mathematics education must be rooted in usefulness, flexibility, and the human activity of making sense, rather than in closed systems delivered for their own sake.

COLLABORATIVE WORK WITH MATHEMATICIANS

The formal world of mathematical thinking constitutes a significant portion of university-level mathematics. Formal mathematics includes both theoretical and axiomatic mathematics, where definitions and deductions are used, while axiomatic mathematics relies solely on set-theoretic definitions and formal proof. These concepts build upon earlier developments in 'conceptual embodiment' and 'operational symbolism' (see Figure 1, Tall, 2013).

Mathematics at the university level, in a metaphorical sense, is like a cold front that mainly covers both the embodied and symbolic worlds. For instance, in undergraduate linear algebra, concepts are often introduced through formal definitions. Students who are typically familiar with embodied and symbolic ways of thinking about mathematics from earlier experiences often initially have trouble understanding that R^2 is not a subspace of R^3. Their prior knowledge from Calculus and Physics courses can lead them to interpret these spaces as 2D and 3D images, such as viewing a plane within the xyz coordinate system, rather than first consulting the formal definition of a subspace. A key point many learners overlook is that a subspace W must be a nonempty subset of a vector space V. While formal mathematics is essential for teaching the subject, introducing the formal world suddenly, like an ice storm, can damage the branches of understanding. This abrupt approach, without meaningful context or appreciation of the embodied and symbolic worlds, can disrupt students' learning process and skills. According to Tall, "As noted in the historical development, formal proof is not the final summit of mathematical thinking" (Tall, 2023, p. 225).

After examining the perspectives of Aristotle, Poincaré, Sierpińska, Tall, and Freudenthal, a common theme emerged: the interaction between practical and theoretical knowledge in mathematics. However, these ideas, while philosophically and historically enriching, must also be tested against the realities of university teaching at the ground level. Collaborative research with mathematicians provides a unique perspective on this process, demonstrating how abstract mathematical theories are applied in practice by experts and students, and how mathematics education theories offer valuable insights. The following four research studies, conducted in collaboration between mathematicians and mathematics educators, may provide additional insights. The studies cover topics in Abstract Algebra (Stewart & Schmidt, 2017), Calculus (Stewart et al., 2015), Algebraic Topology (Stewart, Thompson, & Brady, 2017), and Linear Algebra (Madden, Stewart, & Meyer, 2023).

Accommodation in the formal world

Stewart and Schmidt (2017) examined how mathematician Ralf and his graduate student Kim navigated advanced abstract algebra using Tall's three worlds of mathematical thinking. The study involved analyzing Ralf's daily journals, as well as Kim's, along with audio recordings of weekly research meetings, during a two-semester abstract algebra course taught in reverse order (fields–rings–groups). Key findings revealed major teaching challenges, such as explaining the significance of concepts like Galois Theory, teaching highly abstract ideas, and bridging the gap between mathematical elegance and student struggles. A clear difference in thinking was observed, with Ralf noting that he and Kim felt as if they attended completely different lectures. Ralf's approach focused on remaining within the formal mathematical framework, postponing examples until after proving the main theorem, believing that ‚the air in the formal world is much thinner, but also much clearer.' In contrast, Kim's journal entries initially emphasized learning styles, only showing excitement and including mathematical content after the main theorem was established and the first example was given, noting how the theoretical parts finally connected.

Historically, mathematicians learned about the formal world through examples. In group theory, they understood Galois groups by their action on polynomial zeros, knew the symmetric group through permutations, and saw SO(3) as representing space rotations. Ralf explained embodiment as familiarity with these examples. Each illustrates the general concept of a group and provides intuition. For example, knowing only symmetric groups might suggest all groups are finite, or only SO(3) might imply all groups are infinite or continuous. True understanding emerges from combining these intuitions, stripping away specific details to reveal the core idea. This process is called abstraction; it leads to the group axioms: associativity, identity, and inverses—universally shared properties that enable formal reasoning and theorem proving (Stewart & Schmidt, 2017).

Ralf's description matches Tall's ideas of practical and theoretical thinking, which start with many examples and move toward theories and abstraction. Ralf delayed examples until after proving the main theorem, insisting that "the air in the formal world is much thinner, but also much clearer." Kim, by contrast, only showed excitement once examples were introduced, when "the theoretical parts finally connected." The study highlights the difficulty of initiating students into the formal world. For the expert, elegance lies in the axiomatic structure itself, and for the learner, meaning often arrives only after concrete illustrations. This tension mirrors Aristotle's distinction between knowing "the that" and knowing "the why," and echoes Sierpińska's concern that theory, without careful nurturing, may risk distancing students.

Balancing theory and examples in calculus

Stewart et al. (2015) examined a mathematics instructor's teaching of Calculus I, guided by Tall's model and cognitive psychology's expert–novice distinction. Students valued computations and concrete examples because they aligned with assessments, but often "zoned out" during theoretical introductions. In response, the instructor reversed her lecture order, starting with examples before theory, and frequently used diagrams to support embodied thinking. Yet she noted that visualizations could be double-edged: sometimes copied without understanding, sometimes taken as proof themselves. Her reflections align with Freudenthal's emphasis on mathematizing as an activity and with Poincaré's insistence that mathematics must strike a balance between usefulness and beauty.

Switching between worlds in algebraic topology

In a study of algebraic topology, Stewart, Thompson, and Brady (2017) documented how a geometer (Brady) integrated embodied, symbolic, and formal perspectives when teaching. "When I think of the mathematical world of algebra," he remarked, "all three lenses (embodied, symbolic, formal) kick into gear" (p. 2264). For him, transitions between worlds were seamless. For his students, however, the leap into formal definitions proved most difficult. He responded by guiding them through pictures and intuition before leading them toward proofs. This approach illustrates Tall's view that formal reasoning must grow out of embodied and symbolic foundations, and it underscores the need for pedagogy that respects the learner's trajectory rather than assuming instant access to the formal world.

The language of mathematics

Madden, Stewart, and Meyer (2023) focused on the role of language in teaching linear algebra. Their analysis revealed how an instructor (Meyer) moved between informal speech, geometric gestures, and precise formal definitions within a single lecture. At one point, he asked: "Is a single vector a span? No, right, so we have to go back to our precise definition. So, a single vector is not a span, right, but a line here is a span". Such exchanges illustrate how mathematical meaning emerges not from formalism alone but from the weaving together of spoken words, symbols, and visual references. Tall (2023) emphasized the central role of language in naming and stabilizing concepts, a point already anticipated by Poincaré, who argued that "ordinary language is too poor, too vague, to express relations so delicate".

Making sense of proof

Running through these studies is a central theme: the role of proof. Tall (2023) argued that proof should be meaningful, not a ritual learned by rote. Bayer et al. (2024) pointed out that "there is a striking difference between how a proof is defined in theory and how it is used in practice," a gap increasingly addressed by computer-assisted proof verification. Melhuish et al. (2022), in a synthesis of 104 studies, concluded that we still know little about how specific teaching practices support students' proof activity. One promising effort is described by Meyer, Stewart, and Madden (2024), who designed a second linear algebra course in which students were guided to construct their own proofs through carefully scaffolded resources. The aim was not reproduction but exploration, conjecture, and justification — an approach consistent with Freudenthal's mathematizing, Sierpińska's call to foster theoretical thinking, and Tall's vision of long-term development across worlds.

Challenges of teaching and learning practical and theoretical mathematics

Collectively, these collaborative studies reveal a common theme: the ability to navigate multiple worlds is more achievable for experts than for beginners. Each study highlights the pedagogical challenge of helping learners cross these thresholds, emphasizing strategies such as delaying examples, reversing the order of presentation, combining informal and formal language, or scaffolding proof construction. These insights also resonate with the perspectives of the five thinkers who frame this discussion. Aristotle's emphasis on precise definitions and causes appears in the focus on formal language. Poincaré's call for balancing utility, philosophy, and aesthetics is reflected in struggles to connect abstract elegance with accessible examples. Sierpińska's concern for fostering theoretical thinking is evident in students' difficulties when faced with sudden abstraction. Freudenthal's view of mathematics as an activity of mathematizing, rather than a closed system, aligns with efforts to use pictures, conjectures, and proofs as dynamic processes. Finally, Tall's three-worlds framework provides a unifying perspective, situating these current findings within a broader historical and cognitive context.

Having considered the individual contributions of Aristotle, Poincaré, Sierpińska, Tall, and Freudenthal, it is time to step back and ask what emerges when their ideas are placed side by side. A comparative synthesis helps reveal both convergences and tensions in their views of practical and theoretical mathematics, and sets the stage for educational implications.

COMPARATIVE PERSPECTIVES ACROSS THE FIVE THINKERS

The perspectives of Aristotle, Poincaré, Sierpińska, Tall, and Freudenthal, though developed in different centuries and contexts, can be seen as part of a continuous conversation about the interplay between practical and theoretical mathematics. Each thinker contributed a distinct lens that reflects their historical position and intellectual aims, yet their ideas share resonances that remain relevant for mathematics education today.

Collectively, the views of Aristotle, Poincaré, Freudenthal, Sierpińska, and Tall reveal a shared concern with the relationship between practical and theoretical mathematics. Aristotle distinguished between experience, craft knowledge, and scientific knowledge, ranking them by their depth of explanation. Poincaré, centuries later, demonstrated how mathematics and physics influence each other, with practical problems inspiring bold theoretical progress and theoretical insights sometimes predicting experimental results. Freudenthal redefined mathematics as a human activity, emphasizing its usefulness and the ongoing process of reinvention. Sierpińska highlighted the constant tension in university classrooms, where students tend toward practical methods but need encouragement to develop their theoretical thinking. Tall developed a long-term developmental framework that integrates practical, theoretical, and formal mathematics.

Despite their differences in context and emphasis, all five recognize that knowledge develops through transitions from the particular to the universal, from the concrete to the abstract, from the useful to the systematic. What binds their contributions together is the conviction that both practice and theory are indispensable—and that the challenge lies not in choosing one over the other, but in fostering their mutual enrichment in teaching and learning.

Taken together, these five thinkers highlight different aspects of the challenge of balancing the immediate usefulness and accessibility of mathematics with its deeper, more abstract structures. Aristotle and Poincaré address the philosophical stakes, Freudenthal and Sierpińska reveal the pedagogical dilemmas, and Tall combines these insights into a cohesive developmental framework. Incidentally, Sierpińska (2004) explicitly referenced Aristotle's *Metaphysics*, using his foundational distinction between types of knowledge to shape her model of practical and theoretical mathematical thinking. Their collective contributions point toward a vision of mathematics education that values both practical and theoretical aspects, not as competing goals but as mutually reinforcing parts of mathematical understanding.

At a glance, Table 2 presents the position of each thinker on the relationship between practical and theoretical mathematics, summarizing their key focus in a comparative overview. What we learn is that, across different eras and contexts, the same tension persisted. Furthermore, each thinker provided valuable insights on education.

Table 2. Comparative perspectives on practical and theoretical mathematics

Thinker	Key focus on Knowledge	Practical	Theoretical	Implications for education
Aristotle	Hierarchy of knowledge (*experience → craft → science → wisdom*)	Knowledge of particulars, effective in action	Knowledge of universals and causes, transmissible through teaching	Importance of moving from "knowing that" to "knowing why" in education
Poincaré	Relationship between mathematics and physics	Physics prompts conceptual advances (e.g., Fourier series)	Mathematics anticipates science (e.g., Maxwell, symmetry)	Need to cultivate both utility and beauty in mathematical thinking
Sierpińska	Student engagement with linear algebra	Comfort with computations and examples	Theoretical thinking requires special nurturing	Practical understanding of theory can bridge accessibility and rigor
Tall	Developmental model of mathematical thinking (*three worlds*)	Embodied and symbolic reasoning	Transition to formal axiomatic proof	Pedagogy must support flexible transition across the worlds
Freudenthal	Mathematics as an activity (*mathematizing reality*)	Starts with context and concrete application	Moves toward generalization and reinvention of structures	Teaching should focus on activity, not only on closed systems

The comparative perspectives emphasize the strong link between practice and theory in mathematics. The next question is how these insights can shape university-level instruction.

IMPLICATIONS FOR MATHEMATICS INSTRUCTION AND RESEARCH DIRECTIONS

We now focus our attention on mathematics instruction and endeavor to identify strategies for applying the insights gained from the literature examined in this discussion to the university mathematics classroom. Sierpinska (2000) raised the question: "Is theoretical thinking worthy of our educational efforts?" (p. 245). Using Sierpinska's (2004) model, in a scoping review of 49 linear education studies, Stewart and Troup (2025) conjectured why students may prefer practical thinking over theoretical thinking as follows:

> This scoping review suggests that students may not dislike theoretical thinking inherently, but may be systemically trained to devalue it—they may dislike being passive recipients of it. Education systems could create structural roles that separate theory and practice—roles that students and teachers feel pressured to uphold … Indeed, research suggests that students' theoretical thinking is broader, more present, and more varied than traditional curricula anticipate (Gerami et al., 2024; Kontorovich, 2018; Zandieh et al., 2017). When given the opportunity to explore theory in different ways, students' interest in theoretical thinking increases (Andriunas et al., 2022; Griffiths & Shionis, 2021; McDonald et al., 2024). This suggests the issue may not be students' aversion to theoretical thinking itself but the restrictive ways in which it is often presented, or their conditioned expectation that the assessments will require more practical thinking than theoretical thinking (p. 825).

In their view, efforts to help students develop a practical understanding of theoretical concepts can open up broader opportunities for them (Stewart & Troup, 2025). This approach not only enhances students' comprehension of the material but also equips them with the skills necessary to apply theoretical knowledge in real-world contexts.

In a systematic review, Cevikbas, Kaiser, and Schukajlow (2022) examined 75 papers on mathematical modeling skills and concluded that "modelling and applications are essential components of mathematics, and applying mathematical knowledge in the real world is a core competence of mathematical literacy" (p. 206). Modeling shows the connection between practice and theory: it starts with *practical* work—translating complex real-world situations into mathematical form—but requires *theoretical* thinking to abstract, generalize, and validate models. In this way, modeling is closely linked to problem-solving traditions in physics and engineering, where shifting between context and abstraction is a fundamental principle. However, as Cevikbas et al. point out, the field still lacks a strong theoretical foundation, increasing the risk that modeling becomes context-specific exercises instead of a pathway to the flexible, transferable thinking emphasized by Freudenthal, Tall, and Sierpińska.

Many mathematics instructors believe that students pursuing a mathematics degree need a thorough understanding of theoretical concepts, while students in non-mathematical fields do not require the same level of theoretical depth. Conversely, it is argued that non-majors should focus on practical knowledge, as it is directly relevant to their fields. On the other hand, for mathematics majors, exposure to real-world applications of mathematics is often seen as supplementary rather than essential. This contrast necessitates a closer examination of the educational strategies employed in mathematics curricula, particularly in fostering an appreciation for both theoretical knowledge and practical applications, as both are important and valuable. If the above observation of the education system is indeed accurate, in light of the issues discussed in this paper, it is time to reconsider how we approach mathematics education. As Fruedenthal puts it, “If this means teaching pure mathematics and afterwards to show how to apply it, I am afraid we are no better off. I think this is just the wrong order” (p. 5). As highlighted by Tall (2013, 2023), mathematical understanding evolves through various stages, transitioning from practical embodiment to formal reasoning. Tall (2023) suggested:

> With these formal possibilities in mind, we now return to reflect on how we humans interpret visual information using enactive retinal quality graphics. We will find that it enables us to bridge the transition between practical mathematics and theoretical mathematics in calculus and, if required, to transition further to axiomatic formal analysis (p. 240).

This creates a starting set of key research questions that guide further investigation:

- How do students transition between practical, theoretical, and formal levels of mathematical thinking across different contexts?
- What kinds of tasks, representations, or classroom norms help students move beyond practical manipulation toward generalization, abstraction, and proof, without losing meaning?
- How can educators balance the practical utility of mathematics (problem-solving, modeling, applications) with its theoretical structure (abstraction, deduction, proof) in ways that align with students’ cognitive development and educational needs?
- To what extent does instruction emphasizing the process of mathematizing reality—as an entrance to theory—foster students’ practical understanding of theory, specifically their ability to deploy problem-solving techniques and utilize “smart shortcuts,” compared to instruction that presents mathematics as a “beautiful closed system” derived from axiomatization and systematization?

These questions highlight promising avenues for both teaching and research. By examining how students transition between practical, theoretical, and formal thinking, and by assessing the effects of different teaching methods on these transitions, we can deepen our understanding of what it truly means to teach both practical and theoretical mathematics.

CONCLUDING REMARKS

This paper was inspired by Sierpińska’s challenge regarding whether theoretical thinking is truly valuable to our educational efforts. The answer, drawn from various traditions, is neither simple nor uniform, but it is highly revealing. Aristotle distinguished between experience, craft, and science. Tall suggests that mathematics develops from the practical and embodied to the theoretical and formal. Poincaré demonstrated how mathematics relies not only on logic but also on intuition and application, with theory and practice continually supporting each other. Freudenthal based these insights on pedagogy, viewing mathematics as a human activity where learners reinvent theory through meaningful practice. Tall's developmental model of mathematical thinking (three worlds) laid the groundwork for many of his ideas across all levels of human growth and understanding of mathematics. Tall argued that students' difficulties often stem not from a lack of ability but from the inherently evolving nature of mathematics itself. Sierpińska emphasized that careful nurturing of theoretical thinking is essential, and her model was only an initial step toward understanding a "complex reality". Taking her caution seriously pushes us toward the next stage, which involves developing more refined models that better incorporate epistemological analyses of specific mathematical areas and that can effectively guide didactical practice.

The implications for mathematics education, then, go beyond simply deciding whether theoretical thinking should be taught. The real challenge is how to design instruction that helps students move smoothly between practice and theory, creating spaces where they can mathematize reality, engage with abstraction, and see theory as both a tool and a product of human thought.

Looking ahead, the discussion among philosophy, mathematics, and mathematics education offers a valuable agenda. By examining how mathematical knowledge evolves—and acknowledging its dual role as both practical and theoretical—we can develop teaching methods that respect students' starting points while guiding them toward more advanced reasoning skills. The comparative perspectives of Aristotle, Poincaré, Freudenthal, Sierpińska, and Tall not only illuminate our history but also emphasize the ongoing challenge of creating stronger, more comprehensive educational models of mathematical thinking at the university level.

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